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ON THE INFLUENCE OF GYROSCOPIC FORCES ON THE STABILITY OF STEADY-STATE MOTION

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The question of the influence of gyroscopic forces on the stability of steady-state motion of a holonomic mechanical system when the forces depend upon the velocities of only the position coordinates was answered by the Kelvin-Chetaev theorems [1] on the influence of gyroscopic and dissipative forces on the stability of equilibrium. However, if the gyroscopic forces depend as well on the velocities of the ignorable coordinates, then their influence on the stability of steady-state motions can, as the two problems in [2] show, prove to be entirely different from the influence of gyroscopic forces depending only on the velocities of the position coordinates. In this paper we investigate the influence of gyroscopic forces depending linearly on the velocities of the generalized coordinates, including the ignorable ones, on the stability of the steady-state motion of a holonomic conservative system. We prove that when the gyroscopic forces applied with respect to the ignorable coordinates are given as total time derivatives of certain functions of the position coordinates, the gyroscopic forces can both stabilize as well as destabilize the steady-state motion. Under certain conditions, this influence is also preserved for the action of dissipative forces depending on the velocities of only the position coordinates. In the case of action of dissipative forces depending also on the velocities of the ignorable coordinates, we have indicated the stability and instability conditions of the steady-state motion. Examples are considered. In conclusion, we discuss the conditions under which the application of gyroscopic forces to the system is equivalent to adding terms depending linearly on the generalized velocities to the Lagrange function.

1. We consider a holonomic system with geometric constraints. If the independent Lagrange coordinates q_i and velocities $q_i^\cdot \equiv dq_i/dt$ are taken as the basic variables characterizing the system's state at any instant t, then the system's equations of motion can be written as the Lagrange equations

$$\frac{d}{dt}\frac{\partial L}{\partial q^\cdot_i} - \frac{\partial L}{\partial q_i} = Q_i \quad (i = 1, \ldots, n) \tag{1.1}$$

$$L(q, q^\cdot) = T + U, \qquad T(q, q^\cdot) = \frac{1}{2}\sum_{i,j=1}^{n} a_{ij}(q)\, q_i^\cdot q_j^\cdot$$

Here T is the kinetic energy, $U(q)$ is the force function of the potential forces acting on the system, $Q_i(q, q^\cdot)$ are nonpotential generalized forces. We assume further that the coordinates q_α are ignorable, i.e. the conditions

$$\partial L/\partial q_\alpha = 0 \quad (\alpha = k+1, \ldots, n) \tag{1.2}$$

are satisfied and, in addition, the generalized forces Q_i are independent of the ignorable coordinates. If the nonpotential forces corresponding to the ignorable coordinates, $Q_\alpha = 0$, then Eqs. (1.1) have the first integrals

$$\partial L/\partial q_\alpha^{\cdot} = c_\alpha = \text{const} \quad (\alpha = k+1, \ldots, n) \tag{1.3}$$

By using integrals (1.3) and applying Routh's method for disregarding the ignorable coordinates, the study of the system's motion can be reduced [1] to the integration of a system of $2\ (n-k)$ th-order Routh equations describing the motion of the so-called reduced system, which we call system A, and to subsequent quadratures.

When all the nonpotential forces $Q_i = 0\ (i = 1, \ldots, n)$, under conditions (1.2) and under specified initial conditions Eqs. (1.1) admit of the particular solutions

$$q_s = q_{s0},\ q_s^{\cdot} = 0,\ q_\alpha^{\cdot} = q^{\cdot}_{\alpha 0} \ (s = 1, \ldots, k;\ \alpha = k+1, \ldots, n) \tag{1.4}$$

describing the system's steady-state motion, in which the position coordinates q_s and the velocities $q_\alpha^{\cdot}$ of the ignorable coordinates retain constant values, while the ignorable coordinates q_α vary linearly with time. Constants (1.4) are the solutions of the equations

$$\partial L/\partial q_s = 0 \quad (s = 1, \ldots, k) \tag{1.5}$$

and (1.3) under arbitrarily specified values of constants c_α. The constants $q_{\alpha 0}^{\cdot}$ also can be given arbitrarily; then, allowing for the equalities $q_s^{\cdot} = 0$, from Eqs. (1.5) we find the values of q_{s0}, while from Eqs. (1.3), the values of c_α.

The stability of the steady-state motions can be determined by Routh's theorem and its generalization [3]. An interesting question is that of the influence on the stability of the system's steady-state motions (1.4) of the gyroscopic forces

$$Q_i = \sum_{j=1}^{n} g_{ij} q_j^{\cdot} \quad (i = 1, \ldots, n) \tag{1.6}$$

applied to the system in addition to the potential forces acting on it. The quantities $g_{ij} = -g_{ji}$ are assumed to be continuous functions of the position coordinates $q_1, \ldots, q_k$ possessing continuous first-order partial derivatives in these variables. If forces (1.6) do not depend upon the velocities $q_\alpha^{\cdot}$ of the ignorable coordinates, i.e. all $g_{i\alpha} = 0\ (i = 1, \ldots, n,\ \alpha = k+1, \ldots, n)$, then the forces $Q_\alpha \equiv 0\ (\alpha = k+1, \ldots, n)$. In this case the influence of the gyroscopic forces on the stability of motion (1.4) is completely characterized by the Kelvin-Chetaev theorem [1] on the influence of such forces on the stability of the equilibrium position of the reduced system, corresponding to the steady-state motion of the original system.

In the case of gyroscopic forces depending on $q_\alpha^{\cdot}$, some of the coefficients $g_{\alpha i}$ are nonzero, so that the force $Q_\alpha \not\equiv 0$ and the first integral (1.3) does not correspond to the coordinate q_α. This case, when the Kelvin-Chetaev theorem is not directly applicable, has not been investigated in the literature; meanwhile, the influence of such forces on the steady-state motions' stability can turn out to be entirely different from that of the gyroscopic forces depending on the velocities $q_s^{\cdot}$ of only the position coordinates [2]. Further on we examine this problem, first, under the assumption that the gyroscopic forces Q_α applied to the system along its ignorable coordinates are given as df_α/dt, i.e. as total time derivatives of certain continuous functions $f_\alpha\,(q_1, \ldots, q_k)$, possessing first- and second-order partial derivatives in the position coordinates q_s $(s = 1, \ldots, k)$, and next, for the more general case of forces (1.6).

2. We investigate the influence of gyroscopic forces (1.6) on the stability of some steady-state motion (1.4), assuming that the coefficients

$$g_{\alpha s} = -g_{s\alpha} = \partial f_\alpha/\partial q_s, \quad g_{\alpha\beta} \equiv 0 \quad (s = 1, \ldots, k; \quad \alpha, \beta = k+1, \ldots, n) \tag{2.1}$$

and on the steady-state motion being examined we suppress

$$g_{s\alpha}\,(q_{10}, \ldots, q_{k0}) = 0 \quad (s = 1, \ldots, n; \alpha = k+1, \ldots, n) \tag{2.2}$$

Under conditions (2.2), obviously, Eqs. (1.1) with right-hand sides (1.6) also admit of the solution (1.4) being examined. However, instead of the integrals (1.3), Eqs. (1.1) under conditions (1.2) and forces (1.6) and (2.1) now have the first integrals

$$\partial L/\partial q_\alpha^{\boldsymbol{\cdot}} = f_\alpha\,(q_s) + c_\alpha \quad (\alpha = k+1, \ldots, n) \tag{2.3}$$

Solving Eqs. (2.3) relative to $q_\alpha^{\boldsymbol{\cdot}}$, we find

$$q_\alpha^{\boldsymbol{\cdot}} = \sum_{\gamma=k+1}^{n} b_{\alpha\gamma}(f_\gamma + c_\gamma) - \sum_{s=1}^{k} \gamma_{\alpha s} q_s^{\boldsymbol{\cdot}} \quad (\alpha = k+1, \ldots, n) \tag{2.4}$$

$$b_{\alpha\gamma} = \frac{A_{\gamma\alpha}}{D}, \quad D = \det\,(a_{\alpha\gamma})_{k+1}^{n}, \quad \gamma_{\alpha j} = \sum_{r=k+1}^{n} b_{\alpha r} a_{rj}$$

$A_{\gamma\alpha}$ is the cofactor of element $a_{\gamma\alpha}$ in determinant D. From formulas (2.4) we see that when $f_\alpha\,(q_{s0}) \neq 0$ the values of $q_{\alpha 0}^{\boldsymbol{\cdot}}$ differ from the corresponding values of $q_{\alpha 0}^{\boldsymbol{\cdot}}$ in the absence of forces (1.6) for like values of constants c_α in both cases and, conversely, different values of constants c_α correspond to like values of $q_{\alpha 0}^{\boldsymbol{\cdot}}$.

Let us consider the function defined by the equality

$$R\,(q_s, q_s^{\boldsymbol{\cdot}}, c_\alpha) = L - \sum_{\alpha=k+1}^{n} q_\alpha^{\boldsymbol{\cdot}}\,(f_\alpha + c_\alpha) \tag{2.5}$$

in whose right-hand side $q_\alpha^{\boldsymbol{\cdot}}$ can be replaced by expressions (2.4). Function (2.5) is a Routh function [1] if $f_\alpha = 0$.

It is easy to see that the equalities

$$\frac{\partial L}{\partial q_s} = \frac{\partial R}{\partial q_s} + \sum_{\alpha=k+1}^{n} q_\alpha^{\boldsymbol{\cdot}}\,\frac{\partial f_\alpha}{\partial q_s}, \quad \frac{\partial L}{\partial q_s^{\boldsymbol{\cdot}}} = \frac{\partial R}{\partial q_s^{\boldsymbol{\cdot}}}, \quad q_\alpha^{\boldsymbol{\cdot}} = -\frac{\partial R}{\partial c_\alpha} \tag{2.6}$$

$$(s = 1, \ldots, k, \quad \alpha = k+1, \ldots, h)$$

are valid and, thus, the first k of Eqs. (1.1) with right-hand sides (1.6) become

$$\frac{d}{dt}\frac{\partial R}{\partial q_s^{\boldsymbol{\cdot}}} - \frac{\partial R}{\partial q_s} = Q_s^* \quad (s = 1, \ldots, k) \tag{2.7}$$

where, considering (2.1), the gyroscopic forces

$$Q_s^* = Q_s + \sum_{\alpha=k+1}^{n} q_\alpha^{\boldsymbol{\cdot}}\,\frac{\partial f_\alpha}{\partial q_s} = \sum_{r=1}^{k} g_{sr} q_r^{\boldsymbol{\cdot}} \quad (s = 1, \ldots, k) \tag{2.8}$$

We can investigate the system of Eqs. (2.7) independently of the remaining $n-k$ equations in (1.1). After the integrations of Eqs. (2.7) the variables q_α are found as quadratures from the last group of Eqs. (2.6).

Let us consider the structure of Eqs. (2.7). We see that the function (2.5)

$$R = R_2 + R_1 + R_0$$

$$R_2 = \frac{1}{2}\sum_{s,r=1}^{k} a_{sr}^* q_s^{\cdot} q_r^{\cdot}, \quad a_{sr}^* = a_{sr} - \sum_{\alpha,\gamma=k+1}^{n} b_{\alpha\gamma} a_{\alpha r} a_{\gamma s}$$

$$R_1 = \sum_{s=1}^{k} q_s^{\cdot} \sum_{\alpha=k+1}^{n} \gamma_{\alpha s}(f_\alpha + c_\alpha), \quad R_0 = U - \frac{1}{2}\sum_{\alpha,\gamma=k+1}^{n} b_{\alpha\gamma}(f_\alpha + c_\alpha)(f_\gamma + c_\gamma)$$

Using these equalities we rewrite Eq. (2.7) as

$$\frac{d}{dt}\frac{\partial R_2}{\partial q_s^{\cdot}} - \frac{\partial R_2}{\partial q_s} = \frac{\partial R_0}{\partial q_s} + \sum_{r=1}^{k}(\Gamma_{sr} + g_{sr})\, q_r^{\cdot} \quad (s = 1, \ldots, k) \tag{2.9}$$

Here for brevity we have introduced the notation

$$\Gamma_{sr} = -\Gamma_{rs} = \sum_{\alpha=k+1}^{n}\left[\left(\frac{\partial\gamma_{\alpha r}}{\partial q_s} - \frac{\partial\gamma_{\alpha s}}{\partial q_r}\right)(f_\alpha + c_\alpha) + \frac{\partial f_\alpha}{\partial q_s}\gamma_{\alpha r} - \frac{\partial f_\alpha}{\partial q_r}\gamma_{\alpha s}\right] \quad (s, r = 1, 2, \ldots k)$$

The system with k degrees of freedom, characterized by the Lagrange function (2.5) and forces (2.8), is called reduced system B corresponding to the original system with n degrees of freedom and characterized by the Lagrange function L and the gyroscopic forces (1.6) and (2.1). Allowing for equalities (2.6) and (2.2), it is evident that the equilibrium position $q_s = q_{s0}$ of reduced system B (and also of system A) corresponds to the steady-state motion (1.4) being examined of the original system.

From Eqs. (2.9) we see that the reduced system B is being acted upon by potential (the first summand on the right-hand side of (2.9)) and gyroscopic (the remaining summands) forces, whereas the potential forces $\partial U/\partial q_j$ and the gyroscopic forces (1.6) and (2.1) act on the original system. Thus, the application of gyroscopic forces (1.6) and (2.1) to the original system results in potential and gyroscopic forces, additional in comparison with reduced system A, acting on reduced system B. As we see from the expressions for R_0 and Γ_{ji}, these additional forces depend upon the functions f_α $(q_1, \ldots, q_k)$ defining the gyroscopic forces Q_α applied to the original system along the ignorable coordinates and upon the gyroscopic forces (2.8). Equations (2.7) have the energy integral

$$H(q, q^{\cdot}) = R_2 - R_0 = \text{const}$$

equivalent, in light of (2.4), to the energy integral for Eqs. (1.1) with right-hand sides (1.6).

To investigate the stability of the equilibrium position of reduced system B (A) the Lagrange theorem and the inverse of the Routh theorem, as well as the Kelvin-Chetaev theorem on the influence of gyroscopic and dissipative forces are applicable if the latter depend on the velocities $q_s^{\cdot}$ of the position coordinates only. By comparing the expressions for the part R_0 of function (2.5) for the reduced systems A and B it is easy to see that their difference with like values of constants c_α is

$$\frac{1}{2}\sum_{\alpha,\beta} b_{\alpha\beta}(f_\alpha f_\beta + 2f_\beta c_\alpha)$$

i.e. is a sum of a linear and a quadratic form in f_α. Hence it follows that by a suitable choice of functions $f_\alpha(q_1,\ldots, q_k)$ the function R_0 for system B can be made to have any sign regardless of the sign of the function R_0 for system A. An analogous inference is valid in the case of like values of $q_{\alpha 0}^{\cdot}$. On the basis of the Lagrange theorem and the inverse of the Routh theorem, as well as of the Kelvin-Chetaev theorems, we conclude that the gyroscopic forces (1.6) and (2.1) can have both a stabilizing as well as a destabilizing influence on the steady-state motion (1.4) of the original system, independently of the parity of the degree of stability of reduced system A.

Thus, under known conditions the steady-state motion (1.4) of the original system, unstable (stable) under the action of potential forces, can be stabilized (destabilized) or can be made to remain unstable (stable) by applying suitable gyroscopic forces (1.6) and (2.1) to the system. When dissipative forces, depending on the velocities $q_s^{\cdot}$ of only the position coordinates act on the system, such gyroscopic stabilization is preserved (destroyed) if the function R_0 for system B has an isolated maximum (does not have a maximum and the degree of instability of system B is even); if R_0 does not have a maximum and the degree of stability is odd, the steady-state motion remains unstable. If R_0 has a maximum and the dissipative forces act with dissipation total with respect to $q_s^{\cdot}$ $(s = 1,\ldots, k)$, the perturbed motions asymptotically tend to the steady-state motion corresponding to the maximum of R_0 for the perturbed values of constants c_α [3].

Example 2.1. We consider a heavy rigid body with one fixed point, whose position in an inertial coordinate system is determined by the Euler angles θ, φ, ψ. Assuming that the body's center of gravity is located on one of its principal inertial axes, say, the x-axis, we write the Lagrange equation as

$$L(\theta, \varphi, \theta^{\cdot}, \varphi^{\cdot}, \psi^{\cdot}) = {}^1/_2\,[A(\psi^{\cdot}\sin\theta\sin\varphi + \theta^{\cdot}\cos\varphi)^2 + B(\psi^{\cdot}\sin\theta\cos\varphi - \theta^{\cdot}\sin\varphi)^2 + C(\varphi^{\cdot} + \psi^{\cdot}\cos\theta)^2] - P x_0 \sin\theta\sin\varphi$$

where A, B, C are the principal moments of inertia, P is the weight, x_0 is the coordinate on the body's center of gravity. Equations (1.1) with $Q_i = 0$ have first integrals (1.3) corresponding to the ignorable coordinate ψ and admit of a particular solution of form (1.4)

$$\theta = \varphi = \pi/2, \quad \theta^{\cdot} = \varphi^{\cdot} = 0, \quad \psi^{\cdot} = \omega \quad (c = A\omega) \tag{2.10}$$

describing the permanent rotation with arbitrary angular velocity around a vertically located x-axis. By analyzing the reduced system A it is easy to establish [4] that motion (2.10) is stable with respect to $\theta, \varphi, \theta^{\cdot}, \varphi^{\cdot}, \psi^{\cdot}$ if the conditions

$$a = \frac{c^2}{A^2}(A - C) - Px_0 = (A - C)\omega^2 - Px_0 > 0$$

$$b = \frac{c^2}{A}(A - B) - Px_0 = (A - B)\omega^2 - Px_0 > 0$$

are fulfilled and is unstable if $a < 0, b > 0$ or $a > 0, b < 0$. However, if $a < 0, b < 0$, then by the Kelvin-Chetaev theorems stabilization is possible by forces gyroscopic in the variables $\theta^{\cdot}$ and $\varphi^{\cdot}$ which, however, is destroyed by dissipative forces depending on $\theta^{\cdot}$ and $\varphi^{\cdot}$.

Suppose now that besides the force of gravity the body is also acted upon by gyroscopic forces of form (1.6) and (2.1)

$$Q_1 = k \sin\varphi\,(\varphi^\cdot + \psi^\cdot \cos\theta), \qquad Q_2 = -k\,(\theta^\cdot \sin\varphi - \psi^\cdot \sin\theta\cos\varphi) \qquad (2.11)$$
$$Q_3 = -k\frac{d}{dt}(\sin\theta\sin\varphi)$$

corresponding to the variables $q_1 = \theta$, $q_2 = \varphi$, $q_3 = \psi$, where k - const. Equations (1.1) with right-hand sides (2.11) have a first integral of form (2.3)

$$\partial L/\partial\psi^\cdot = -k\sin\theta\sin\varphi + c$$

from which we find

$$\psi^\cdot = [c - k\sin\theta\sin\varphi - (A-B)\,\theta^\cdot \sin\theta\sin\varphi\cos\varphi - C\varphi^\cdot\cos\theta] \times$$
$$[(A\sin^2\varphi + B\cos^2\varphi - C)\sin^2\theta + C]^{-1}$$

The integration constant $c = A\omega + k$ for motion (2.10). The Lagrange function (2.5) for reduced system B is

$$R(\theta, \varphi, \theta^\cdot, \varphi^\cdot, c) = {}^1/_2\{[AB\sin^2\theta + (A\cos^2\varphi + B\sin^2\varphi)C\cos^2\theta]\,\theta^{\cdot 2} +$$
$$(A\sin^2\varphi + B\cos^2\varphi)C\varphi^{\cdot 2}\sin^2\theta + 2(c - k\sin\theta\sin\varphi) \times$$
$$[(A-B)\,\theta^\cdot\sin\theta\sin\varphi\cos\varphi + C\varphi^\cdot\cos\theta] -$$
$$2(A-B)C\theta^\cdot\varphi^\cdot\sin\theta\cos\theta\sin\varphi\cos\varphi - (c - k\sin\theta\sin\varphi)^2\} \times$$
$$[(A\sin^2\varphi + B\cos^2\varphi - C)\sin^2\theta + C]^{-1} - Px_0\sin\theta\sin\varphi$$

The gyroscopic forces (2.8) are

$$Q^*_1 = k\sin\varphi\varphi^\cdot, \qquad Q^*_2 = -k\sin\varphi\theta^\cdot$$

By computing the second variation of function R_0 we find the stability coefficients for system B

$$\lambda_1 = -Px_0 + \frac{c-k}{A^2}[(A-C)(c-k) + Ak] = -Px_0 + (A-C)\,\omega^2 + \omega k$$

$$\lambda_2 = -Px_0 + \frac{c-k}{A^2}[(A-B)(c-k) + Ak] = -Px_0 + (A-B)\,\omega^2 + \omega k$$

Consequently, the inequalities

$$\lambda_1 > 0, \qquad \lambda_2 > 0 \qquad (2.12)$$

are sufficient conditions for stability with respect to variables $\theta, \varphi, \theta^\cdot, \varphi^\cdot, \psi^\cdot$ of motion (2.10) of the rigid body under the force of gravity. However, if

$$\lambda_1 < 0, \ \ \lambda_2 > 0 \quad \text{or} \quad \lambda_1 > 0, \ \ \lambda_2 < 0 \qquad (2.13)$$

motion (2.10) is unstable. Stabilization by forces gyroscopic in the variables $\theta^\cdot$ and $\varphi^\cdot$ is possible if $\lambda_1 < 0, \lambda_2 < 0$.

With dissipative forces depending upon $\theta^\cdot$ and $\varphi^\cdot$, stability (instability) is preserved under conditions (2.12),(2.13); if the dissipative forces possess total dissipation, then under their action the perturbed motions with conditions (2.12) asymptotically tend to the permanent rotation corresponding to the maximum of R_0 for a perturbed value of constant c, while stability is destroyed under the conditions $\lambda_1 < 0$, $\lambda_2 < 0$. By comparing the expressions for λ_i with the expressions for a and b, we see that by a suitable choice of the magnitude and sign of constant k we can satisfy conditions (2.12) or (2.13) independently of the signs of a and b.

3. We consider the influence on the stability of a certain steady-state motion (1.4) of the forces

$$Q_i = \sum_{j=1}^{n} g_{ij} q_j^{\cdot} - \frac{\partial \varphi}{\partial q_i^{\cdot}} + F_i \quad (i = 1, \ldots, n) \tag{3.1}$$

being superpositions of gyroscopic forces (1.6), of dissipative forces $-\partial\varphi/\partial q_i^{\cdot}$, namely, the derivatives of the dissipative nonnegative Rayleigh function

$$2\varphi(q^{\cdot}) = \sum_{i,j=1}^{n} \beta_{ij} q_i^{\cdot} q_j^{\cdot} \quad (\beta_{ij} = \beta_{ji} = \mathrm{const}) \ .$$

and of forces F_i, constant in magnitude and direction, determined by the equations

$$\sum_{\alpha=k+1}^{n} g_{i\alpha}(q_{s0}) q_{\alpha 0}^{\cdot} - \sum_{\alpha=k+1}^{n} \beta_{i\alpha} q_{\alpha 0}^{\cdot} + F_i = 0 \quad (i = 1, \ldots, n) \tag{3.2}$$

In the considered steady motion the application of constant forces F_i to the system in addition to gyroscopic and dissipative forces is sufficient [5] for balancing the dissipative and, when conditions (2.2) are not satisfied, also the gyroscopic forces.

We examine simultaneously both classes of gyroscopic forces (1.6); the special class when conditions (2.1) are satisfied, as well as the more general one when the conditions

$$g_{\alpha s} = \partial f_\alpha / \partial q_s + \overset{*}{g}_{\alpha s}, \ \overset{*}{g}_{\alpha s} = -\overset{*}{g}_{s\alpha} \quad (s = 1, \ldots, k, \ \alpha = k+1, \ldots, n),$$

are fulfilled, assuming only the satisfaction of the conditions

$$(\partial f_\alpha / \partial q_s)_0 = 0, \quad g_{i\alpha}^{*} = \mathrm{const} \quad (i = 1, \ldots, n) \tag{3.4}$$

Equations (1.1) with right-hand sides (3.1) do not have integrals of form (1.3) or (2.3), nor an energy integral if $\varphi(q^{\cdot}) \not\equiv 0$; however, they do admit of the energy equation

$$\frac{dH}{dt} = \sum_{i=1}^{n} F_i q_i^{\cdot} - 2\varphi(q^{\cdot}) \tag{3.5}$$

Instead of variables $q_\alpha^{\cdot}$, under conditions (1.2) and with the generalized forces Q_i independent of the ignorable coordinates, it is expedient to examine the variables

$$p_\alpha = \partial L / \partial q_\alpha^{\cdot} - f_\alpha(q_s) \quad (\alpha = k+1, \ldots, n) \tag{3.6}$$

In the case of the general class of forces (1.6), when the conditions $g_{i\alpha} = g_{i\alpha}^{*} = \mathrm{const}$ are satisfied, we should set $f_\alpha(q_s) \equiv 0$ in equalities (3.6) and in those obtained with their aid. By solving Eqs. (3.6) relative to $q_\alpha^{\cdot}$, we obtain equalities of form (2.4), in which, as in the preceding formulas, we should merely replace the constants c_α by the variables p_α, doing which we represent the system's energy as

$$H(q_s, q_s^{\cdot}, p_\alpha) = R_2 - R_0 = \frac{1}{2} \sum_{i,j=1}^{k} \overset{*}{a}_{ij} q_i^{\cdot} q_j^{\cdot} + \frac{1}{2} \sum_{\alpha,\beta=k+1}^{n} b_{\alpha\beta}(p_\alpha + f_\alpha)(p_\beta + f_\beta) - U$$

For steady-state motion (1.4) the quantities $p_\alpha = p_{\alpha 0} = \mathrm{const}$. In the perturbed motion we set

$$q_s = q_{s0} + x_s, \quad p_\alpha = p_{\alpha 0} + \eta_\alpha \tag{3.7}$$

and we represent the energy in a neighborhood of motion (1.4) as

$$H(q_{s0} + x_s, \dot{x}_s, p_{\alpha 0} + \eta_\alpha) = (H)_0 + \sum_{\alpha=k+1}^{n} q_{\alpha 0}^{\cdot} \eta_\alpha + H^{(2)}(x_s, x_s^{\cdot}, \eta_\alpha) + \ldots$$

where $H^{(2)}(x_s, x_s^{\cdot}, \eta_\alpha)$ is the second variation of the energy, $q_{\alpha 0}^{\cdot} = -(\partial R/\partial p_\alpha)_0$, the symbol $(a)_0$ denotes the value of quantity a on motion (1.4), and the dots denote terms of the order higher than second in $x_s, x_s^{\cdot}, \eta_\alpha$.

From Eq. (3.5) with due regard to Eqs. (3.2) and (3.4) and from the equations for the variables p_α, obtained from (1.1) with due regard to (3.1) and (3.6), follows the equation

$$\frac{d}{dt}[H^{(2)}(x_s, x_s^{\cdot}, \eta_\alpha) + \ldots] = -\sum_{i,j=1}^{n} \beta_{ij} x_i^{\cdot} x_j^{\cdot} \tag{3.8}$$

in whose right-hand side we can replace the variables $x_\alpha^{\cdot}$ by their expressions computed with equalities (3.7) accounted for

$$x_\alpha^{\cdot} = \left(\frac{\partial R}{\partial p_\alpha}\right)_0 - \frac{\partial R}{\partial p_\alpha}$$

By virtue of the general theorems of Liapunov's second method [1, 6], on the basis of Eq. (3.8) we obtain the following conclusion on stability.

If the dissipative forces possess total dissipation, then the steady-state motion (1.4) being examined is asymptotically stable with respect to the variables $q_s, q_s^{\cdot}, p_\alpha$ ($s = 1, \ldots, k$; $\alpha = k+1, \ldots, n$) when the second variation $H^{(2)}(x_s, x_s^{\cdot}, \eta_\alpha)$ of the system's energy is a positive definite function of the variables $x_s, x_s^{\cdot}, \eta_\alpha$, and is unstable when $H^{(2)}(x_s, x_s^{\cdot}, \eta_\alpha)$ can take negative values for values of $x_s, x_s^{\cdot}, \eta_\alpha$ arbitrarily small in modulus. These same results are valid for dissipative forces with partial dissipation if the set

$$\sum_{ij=1}^{n} \beta_{ij} x_i^{\cdot} x_j^{\cdot} = 0$$

does not contain other whole motions of the system besides the motion $x_s = x_s^{\cdot} = 0$, $\eta_\alpha = 0$. If the dissipative forces are absent or possess partial dissipation, while the function $H^{(2)}(x_s, x_s^{\cdot}, \eta_\alpha)$ is positive definite, then the steady-state motion (1.4) is stable with respect to the variables $q_s, q_s^{\cdot}, p_\alpha$ ($s = 1, \ldots, k$; $\alpha = k+1, \ldots, n$).

Example 3.1. We continue the consideration of Example 2.1, assuming that in addition to the gyroscopic forces (2.11) the dissipative forces, the derivatives of the positive definite function

$$2\varphi = \sum_{i,j=1}^{3} \beta_{ij} q_i^{\cdot} q_j^{\cdot} \quad (\beta_{ij} = \beta_{ji} = \text{const})$$

where the q_i are the Euler angles θ, φ, ψ, and constant forces $F_i = \beta_{i3}\,\omega$ ($i = 1, 2, 3$), also act on the heavy rigid body. For this problem the second variation of energy

$$2H^{(2)}(x_1, x_2, x_1^{\cdot}, x_2^{\cdot}, \eta) = Bx_1^{\cdot 2} + Cx_2^{\cdot 2} + \frac{1}{A}\eta^2 + \\ [(A - C)\omega^2 + \omega k - Px_0]\,x_1^2 + [(A - B)\omega^2 + \omega k - Px_0]\,x_2^2$$

is positive definite under conditions (2.12) and is sign variable under conditions (2.13) as well as for $\lambda_1 < 0$, $\lambda_2 < 0$. Consequently, the permanent rotation (2.10) of the heavy rigid body under gyroscopic forces (2.11) and dissipative forces with total dissipation is asymptotically stable with respect to the variables $\theta, \varphi, \theta^{\cdot}, \varphi^{\cdot}, \psi^{\cdot}$, if conditions (2.12) are fulfilled, and is unstable under conditions (2.13) or when $\lambda_1 < 0$, $\lambda_2 < 0$. For example, the rotation of a Kowalewska top ($A = B = 2C$) around the vertical is asymptotically stable if $\omega k - Px_0 > 0$ and is unstable if $\omega k - Px_0 < 0$.

4. In conclusion, we dwell briefly on the existence of a generalized potential for gyroscopic forces (1.6). If the Lagrange function L contains the terms

$$L_1 = \sum_{i=1}^{n} a_i(q_1, \ldots, q_n) q_i^{\cdot} \tag{4.1}$$

linear in the generalized velocities $q_i^{\cdot}$, where the $a_i(q_1, \ldots, q_n)$ are assumed to be differentiable functions, and if the function $L_1 dt$ is not equal to the total differential of any function of variables q_i, then (see [7], for example) in the Lagrange equations (1.1) to these terms correspond gyroscopic forces, the derivatives of function L_1

$$Q_i = -\frac{d}{dt}\frac{\partial L_1}{\partial q_i^{\cdot}} + \frac{\partial L_1}{\partial q_i} = \sum_{j=1} g_{ij} q_j^{\cdot} \quad (i = 1, \ldots, n) \tag{4.2}$$

Here the coefficients $g_{ij} = -g_{ji}$ are determined by the equalities

$$g_{ij} = \partial a_j / \partial q_i - \partial a_i / \partial q_j \quad (i, j = 1, \ldots, n) \tag{4.3}$$

In particular, if $\partial L_1 / \partial q_\alpha \equiv 0$, then

$$g_{\alpha s} = -\partial a_\alpha / \partial q_s, \quad g_{\alpha\beta} = 0 \quad (s = 1, \ldots, k;\ \alpha, \beta = k+1, \ldots, n)$$

and

$$Q_\alpha = \sum_{j=1}^{n} g_{\alpha j} q_j^{\cdot} = -\frac{da_\alpha(q_1 \ldots q_k)}{dt} \tag{4.4}$$

The reverse statement is valid under definite conditions on the coefficients $g_{ij}(q_1, \ldots, q_n)$ in expressions (1.6): the application of gyroscopic forces (1.6) to the system is equivalent to adding a certain function L_1 of form (4.1), linear in the generalized velocities $q_i^{\cdot}$, to the Lagrange function L. In fact, for this it is necessary and sufficient that the forces (1.6) applied to the system have a generalized force function L_1, i.e. could be represented as the mean parts of equalities (4.2), but this is possible if and only if the first-order partial differential Eqs. (4.3), whose left-hand sides are given functions $g_{ij}(q) = -g_{ji}(q)$, are compatible and have a solution.

The number $n(n-1)/2$ of solutions of Eqs. (4.3) does not equal the number n of unknown functions $a_i(q)$ except when $n = 3$; when $n = 2$ there is only one equation and when $n > 3$ the number of equations is greater than the number of unknowns by $n(n-3)/2$. For the complete integrability of Eqs. (4.3) it is necessary and sufficient that they be compatible. Assuming that the given functions $g_{ij}(q)$ are of class C^1, while the unknown functions $a_i(q)$ are of class C^2, it is not difficult to obtain the conditions for the compatibility of Eqs. (4.3) in the form

$$\partial g_{ij} / \partial q_r - \partial g_{rj} / \partial q_i = \partial g_{ir} / \partial q_j \quad (i, j, r = 1, \ldots, n) \tag{4.5}$$

When conditions (4.5) are fulfilled the gyroscopic forces (1.6) have a generalized force function of form (4.1). The determination of the functions $a_i(q)$ is reduced to ordinary differential equations [8].

If $g_{ij} = -g_{ji} = \text{const}$, conditions (4.5) are always fulfilled and Eqs. (4.3) have the solution

$$a_i(q) = -\frac{1}{2}\sum_{j=1}^{n} g_{ij} q_j + c_i \quad (i = 1, \ldots, n)$$

(c_i are arbitrary constants). If the functions $g_{ij} = g_{ij}(q_1, \ldots, q_k)$ do not depend upon the coordinates q_α and if $g_{\alpha\beta} = 0$ ($\alpha, \beta = k+1, \ldots, n$), then we can seek the unknowns a_i which also do not depend upon q_α. The Eqs. (4.3) for a_α and the conditions (4.5) for their compatibility take, respectively, the form

$$\frac{\partial a_\alpha}{\partial q_s} = g_{s\alpha}, \quad \frac{\partial g_{s\alpha}}{\partial q_r} = \frac{\partial g_{r\alpha}}{\partial q_s} \quad (s, r = 1, \ldots, k, \alpha = k+1, \ldots, n)$$

and if the functions $g_{s\alpha}$ satisfy these conditions, then

$$a_\alpha (q_1, \ldots, q_k) = \int \sum_{i=1}^{k} g_{i\alpha} dq_i + c_\alpha \quad (\alpha = k+1, \ldots, n) \qquad (4.6)$$

We note that in this case the gyroscopic forces Q_α have the form (4.4). Thus, for forces (4.4) have the generalized force function

$$L_1 = \sum_{\alpha=k+1}^{n} a_\alpha (q_1, \ldots, q_k) \dot{q}_\alpha$$

Comparing expressions (4.4) with (1.6) and (2.1), we see that $a_\alpha = -f_\alpha + \text{const}$, which, by the way, is reflected in the form of function (2.5).

Example 4.1. For the gyroscopic forces (2.11) of Example 2.1

$$g_{12} = k \sin \varphi, \; g_{13} = k \cos \theta \sin \varphi, \; g_{23} = k \sin \theta \cos \varphi$$

The function $a_3 (\theta, \varphi)$ is found by formula (4.6)

$$a_3 (\theta, \varphi) = \int (g_{13}\, d\theta + g_{23}\, d\varphi) = k \sin \theta \sin \varphi + c$$

There is only one Eq. (4.3) with $i = 1$ and $j = 2$ for the functions $a_i (\theta, \varphi)$ $(i = 1, 2)$ in view of which there is much arbitrariness in their determination. As function a_1 we can take, for example, an arbitrary function $a_1 (\theta, \varphi)$; then we find function a_2 as

$$a_2 (\theta, \varphi) = \int \frac{\partial a_1}{\partial \varphi} d\theta + k\theta \sin \varphi + \chi_2 (\varphi)$$

Thus, if $a_1 = k \sin \theta \sin \varphi + \chi_i (\theta)$, then $a_2 (\theta, \varphi) = k (\theta \sin \varphi - \cos \theta \cos \varphi) + \chi_2 (\varphi)$, where the χ_i are arbitrary functions.

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